Modeling loop performance

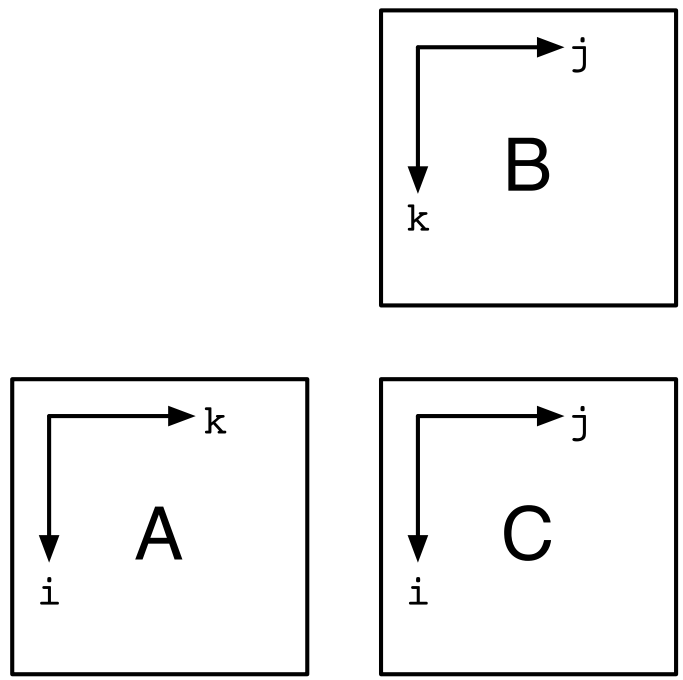
Loop transformations

- Many kinds of loop transformations
  - Loop permutation/interchange
  - Loop blocking/tiling
  - Loop reversal
  - Loop fusion
- Want to understand the effects of these transformations
  - How does a transformation impact performance?
  - Can we predict this impact?
- Focus on a case study: matrix-matrix multiply and loop interchange

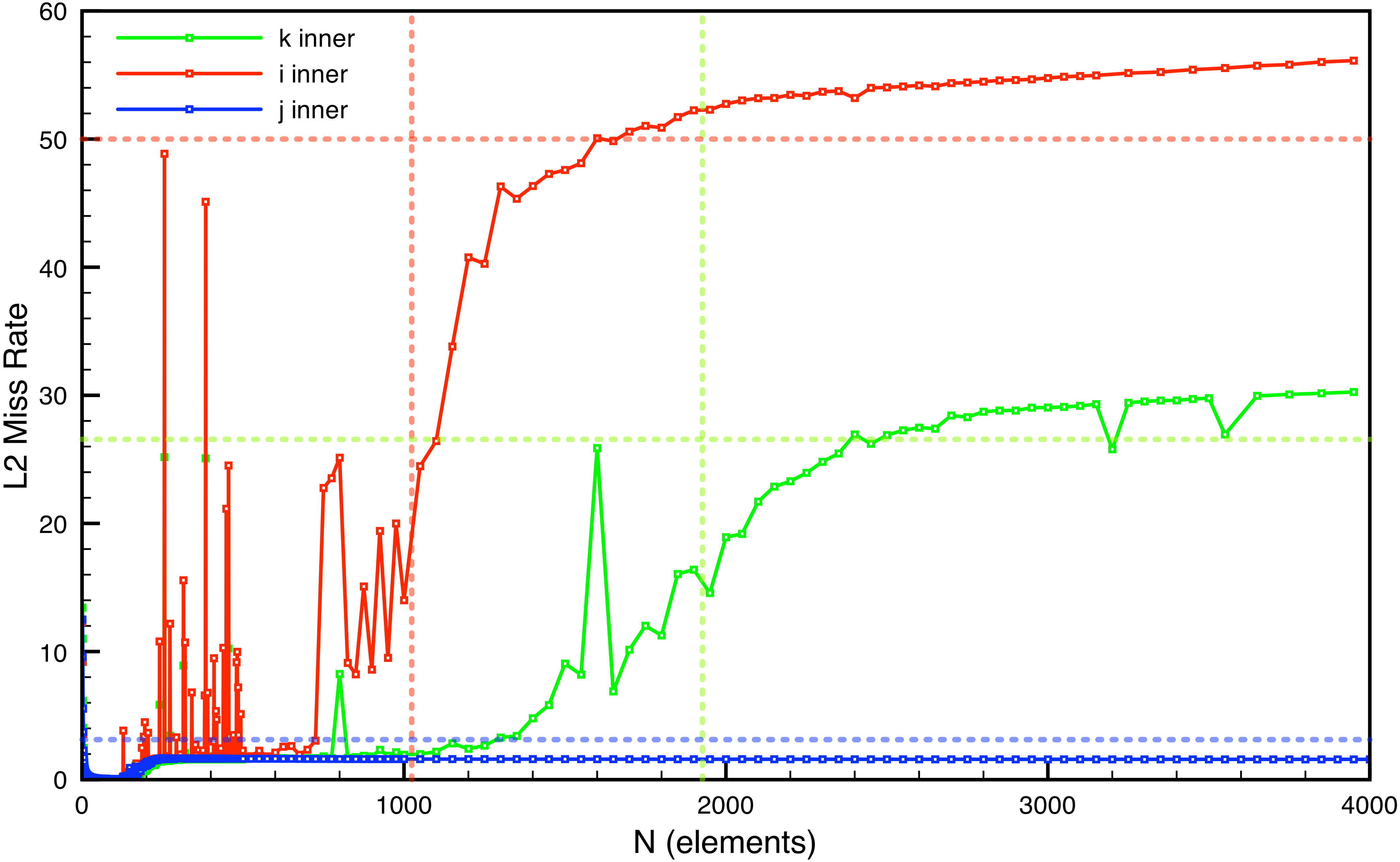
Matrix-matrix multiply

- Key kernel in linear algebra
- How much data? How much computation?
    - Significant data reuse
- Important factor in performance: miss ratio
    - Does miss ratio depend on problem size?
    - Interesting fact: can execute loops in any order
        - Does miss ratio depend on loop order?
- Can we predict miss ratio?

$$
\begin{array}{l}
\texttt{for } i \in [0:1:N-1] \\
\quad \texttt{for } j \in [0:1:M-1] \\
\quad\quad \texttt{for } k \in [0:1:K-1] \\
\quad\quad\quad \mathsf{C}_{ij} \;=\; \mathsf{C}_{ij} \;+\; \mathsf{A}_{ik} \;\star\; \mathsf{B}_{kj}
\end{array}
$$

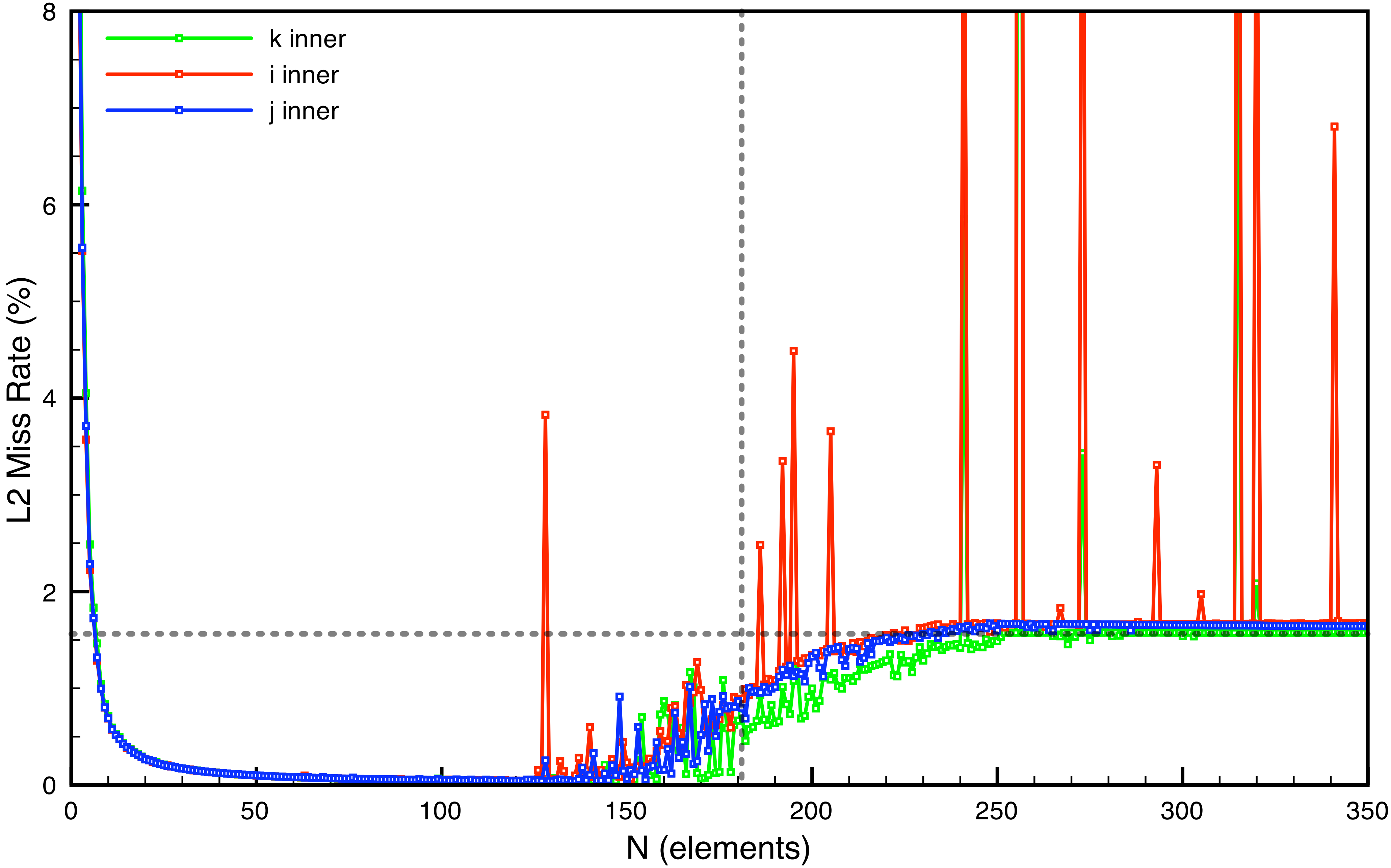


Miss ratios



k inner
i inner
j inner
L2 Miss Rate
N (elements)
0
10
20
30
40
50
60
0
1000
2000
3000
4000

Miss ratios



k inner
i inner
j inner
L2 Miss Rate (%)
N (elements)
0
2
4
6
8
0
50
100
150
200
250
300
350

Explaining miss ratios

- When matrices are small, everything fits in cache
    - Only get cold misses, no capacity misses
    - Misses: $3N^2/b$, accesses: $4N^3$ (why?)
    - Miss rate: $3/(4bN)$
    - Miss rate goes down as problem size goes up!
- How long does this happen?
    - Naive guess: happens as long as all three matrices fit in cache ($N \le \mathrm{sqrt}(C/3)$)
    - On Itanium: should happen when $N \le 104$

Miss-rate regimes

- When a matrix fits entirely in cache, it experiences temporal and spatial locality
  - Misses: N^2/b
- If a matrix is being walked in row-major order, it may experience spatial locality, but not temporal locality
  - Only get a miss 1 out of b accesses
  - Misses: N^3/b
- Other times, a matrix experiences no locality
  - Every access misses
  - Misses: N^3
- (What about a matrix that only experiences temporal locality?)

Predicting miss rates

- To predict a miss rate, we need to determine, for each matrix:
  - Whether it experiences no locality, spatial locality, or both spatial and temporal locality
  - At which point the matrix transitions between the various regimes

Stack distance

- Introduced by Mattson *et al.* in 1970
- The *stack distance* of a memory location is the number of distinct cache lines touched between successive accesses to that location
  - Called stack distance because it can be calculated with a reuse stack
  - Also called *reuse distance*

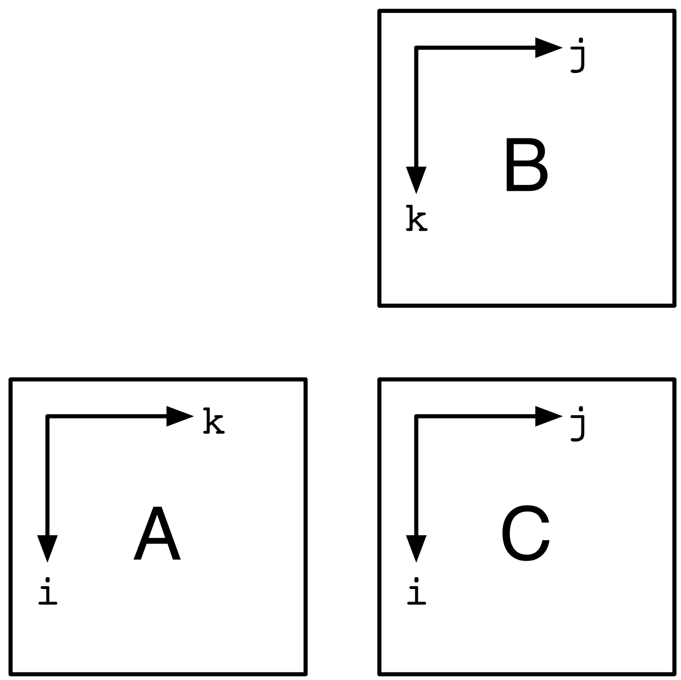
Stack distance

- We introduce two types of stack distance:
  - $d_t(M)$: the stack distance between successive accesses to a given element of M
  - $d_s(M)$: the *minimum* stack distance between successive accesses to distinct elements of M that lay on the same cache line
- If $C < b^* d_t(M)$, matrix does not have temporal locality
  - By the time we touch the same element again, we've brought in too many other elements into cache
- If $C < b^* d_s(M)$, matrix does not have spatial locality

Computing d_t and d_s

for $i \in [0:1:N-1]$
  for $j \in [0:1:M-1]$
    for $k \in [0:1:K-1]$
      $C_{ij} = C_{ij} + A_{ik} \star B_{kj}$

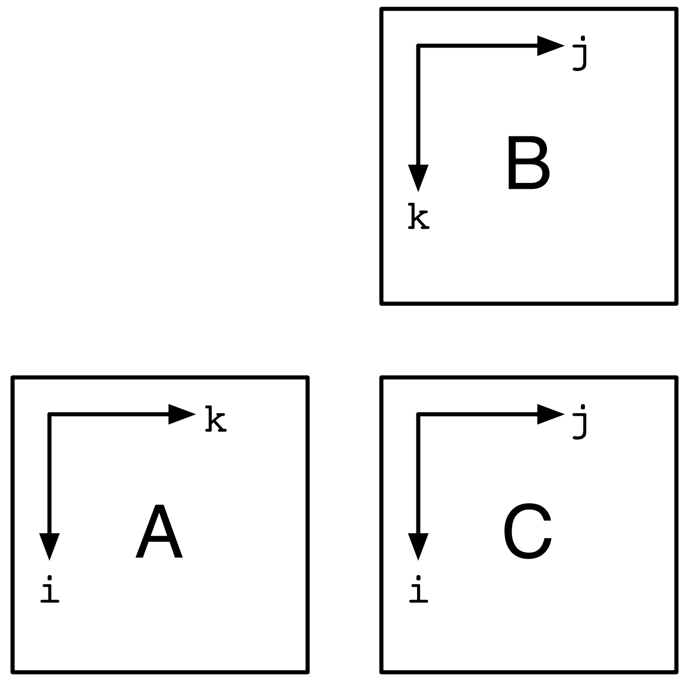
- A is walked in row major order in inner loop
  - d_s(A) = 3 (why?)
  - Note: d_s is not dependent on N → A always has spatial locality
- How many iterations does it take to return to the same element of A?
- What is d_t(A)?



Miss rates for B and C

$$\texttt{for } i \in [0:1:N-1]$$
$$\quad \texttt{for } j \in [0:1:M-1]$$
$$\quad\quad \texttt{for } k \in [0:1:K-1]$$
$$\quad\quad\quad \mathsf{C}_{ij} = \mathsf{C}_{ij} + \mathsf{A}_{ik} \star \mathsf{B}_{kj}$$

- What is d_t(B)?
- What is d_s(B)?
- What about for C?



Putting it all together

$$miss_{ijk,A}(N,b,C) = \left\{ \begin{array}{l|l} N^2/b & bN + N \leq C \\ N^3/b & \text{otherwise} \end{array} \right.$$

$$miss_{ijk,B}(N,b,C) = \left\{ \begin{array}{l|l} N^2/b & (N^2 + 2N) \leq C \\ N^3/b & bN + N \leq C \\ N^3 & \text{otherwise} \end{array} \right.$$

$$miss_{ijk,C}(N,b,C) = N^2/b$$

Putting it all together

$$miss_{ijk}(N,b,C) = \begin{cases} 3N^2/b & (N^2+2N) \leq C \\ N^3/b + 2N^2/b & bN + N \leq C \\ (b+1)N^3/b & \text{otherwise} \end{cases}$$

$$ratio_{ijk}(N,b,C) = \begin{cases} 3/(4bN) & N \leq \sqrt{(}C) \\ 1/(4b) & N \leq C/(b+1) \\ (b+1)/(4b) & \text{otherwise} \end{cases}$$

Miss ratios for other orders

$$ratio_{ijk}(N,b,C) = \left\{ \begin{array}{l|l} 3/(4bN) & N \leq \sqrt(C) \\ 1/(4b) & N \leq C/(b+1) \\ (b+1)/(4b) & \text{otherwise} \end{array} \right.$$

$$ratio_{jki}(N,b,C) = \left\{ \begin{array}{l|l} 3/(4bN) & N \leq \sqrt(C) \\ 1/(4b) & N \leq C/(2b) \\ 1/2 & \text{otherwise} \end{array} \right.$$

$$ratio_{kij}(N,b,C) = \left\{ \begin{array}{l|l} 3/(4bN) & N \leq \sqrt(C) \\ 1/(4b) & N \leq C/2 \\ 1/(2b) & \text{otherwise} \end{array} \right.$$

Performance regimes

- *Large-cache* regime
  - All matrices exhibit temporal and spatial locality
  - Miss rate decreases as matrix size gets larger
  - For all orders, occurs until $N \geq \mathrm{sqrt}(C)$
- *Medium-cache* regime
  - One matrix starts incurring capacity misses
  - Others still enjoy locality
- *Small-cache* regime
  - Two matrices start suffering capacity misses

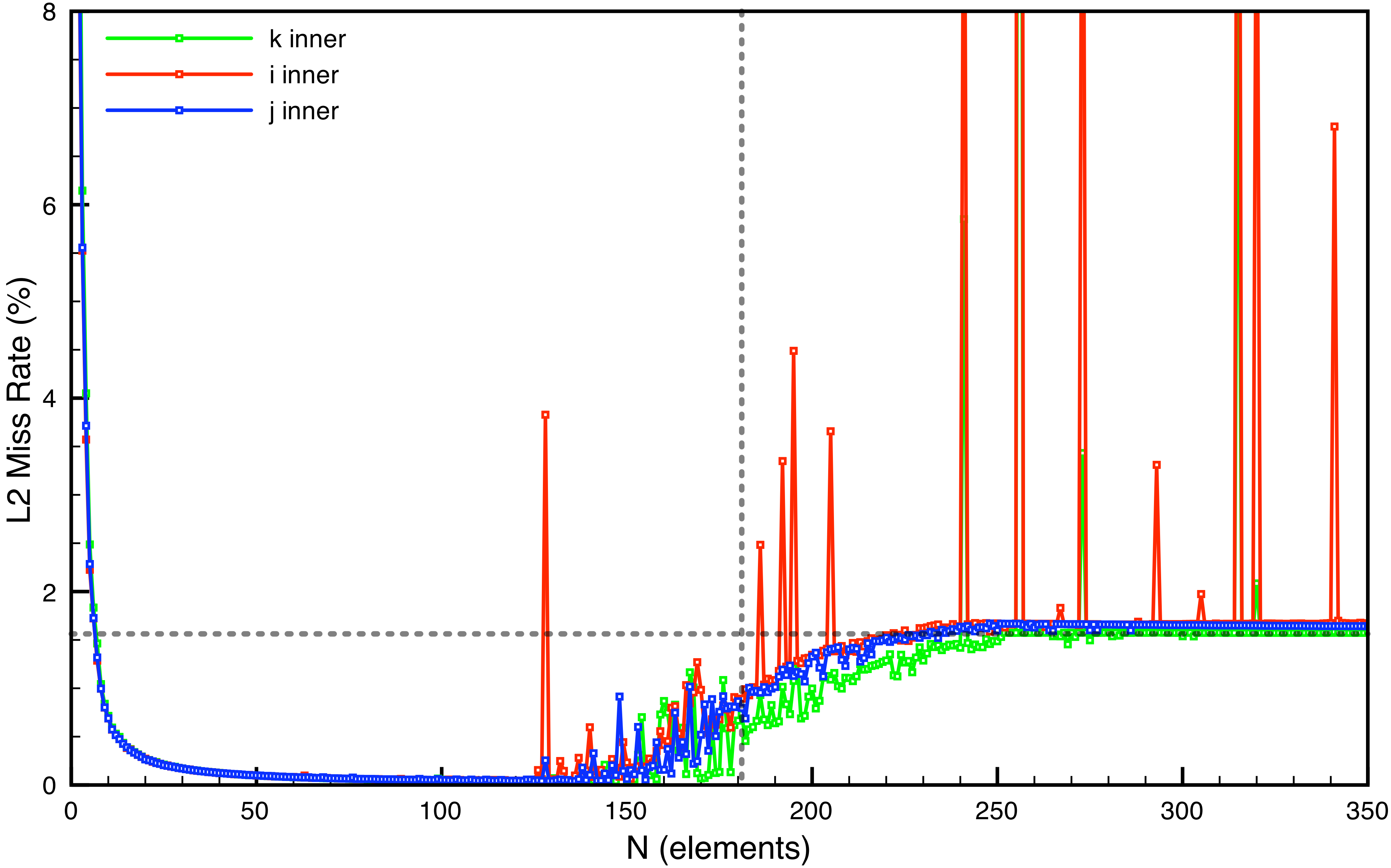
Predicting performance

- Itanium 2 architecture:
  - Line size: 16 doubles
  - Cache size (L2): 32K doubles

$$ratio_{ijk}(N,b,C) = \begin{cases} 3/(4bN) & N \leq \sqrt(C) \\ 1/(4b) & N \leq C/(b+1) \\ (b+1)/(4b) & \text{otherwise} \end{cases}$$

- Predicted miss rates:
  - decrease while N ≤ 181
  - 1.5625% while N ≤ 1927
  - 26.5625% afterwards

Miss ratios



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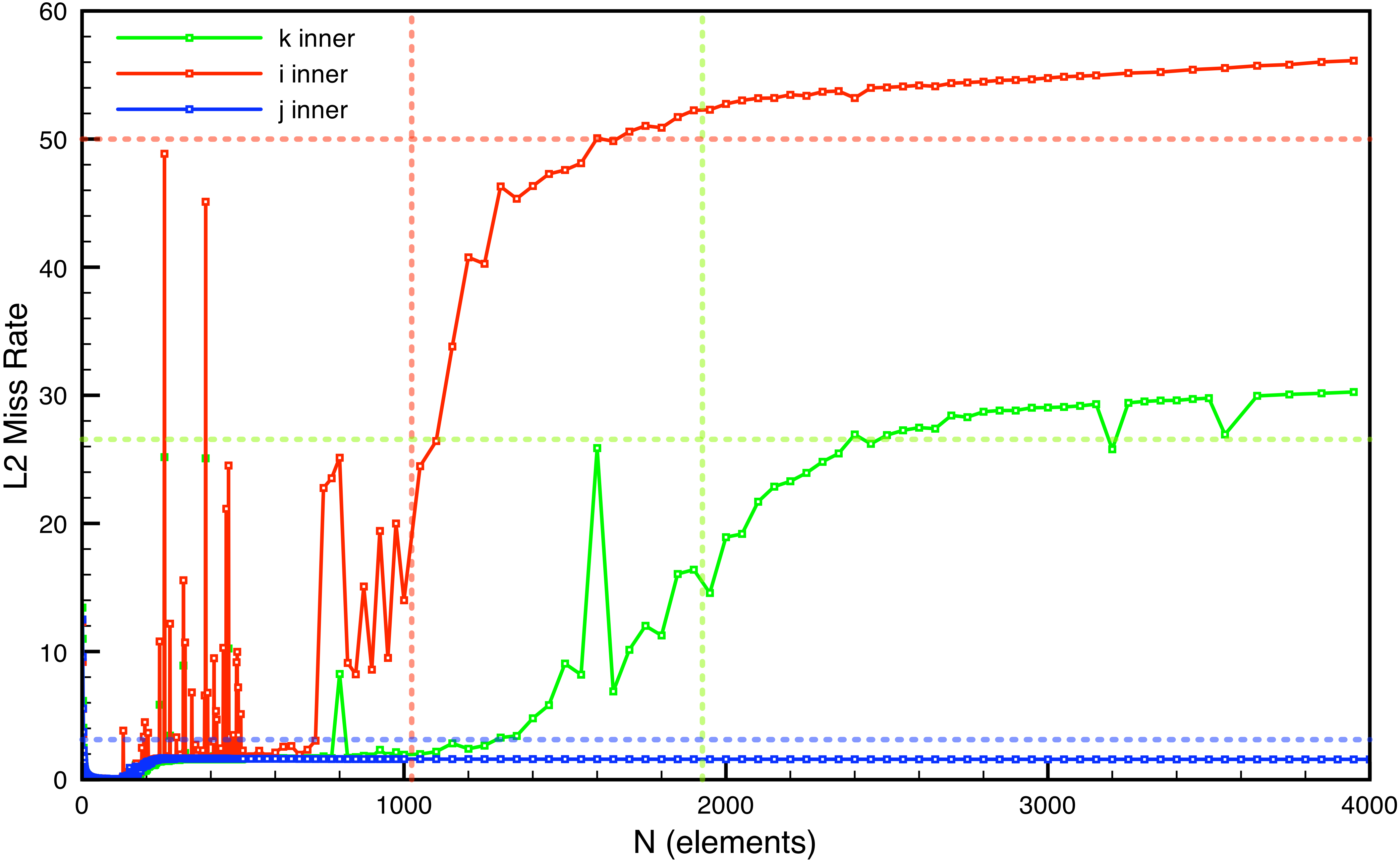
Miss ratios



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Loop permutation

- Loop permutation clearly an important transformation
  - Can lead to massive performance improvements
- How do we determine when loop permutation is legal?
- How do we automatically generate permuted code?
  - Straightforward for some loops (like MMM)
  - Much harder for other loops
- How do we know if loop permutation will be useful?
  - Don't want to change *ikj* loop into *jki* loop!
- Are there other transformations we might want to perform?
- The next set of lectures will answer these questions